Name of College – S.S. College, J'bad

Deptt – Mathematics

Topic – problem based on Tangent and Normal (Contd)

Class – B.Sc I (Hons)

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Study Material

problem :→ If the tangent at (x_1, y_1) to the curve $x^3 + y^3 = a^3$ meets the curve again in (x_2, y_2) show that

$$\frac{x_2}{x_1} + \frac{y_2}{y_1} = -1$$

Solution :→ By the question,

Equation of the curve

$$x^3 + y^3 = a^3$$

Diff. with respect to x

$$3x^2 + 3y^2 \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^2}{y^2}$$

∴ Equation of tangent at (x_1, y_1)

$$y - y_1 = \cancel{-\frac{x^2}{y^2}(x_1, y_1)}$$

$$= -\frac{x_1^2}{y_1^2}(x - x_1)$$

As the Equation of tangent is $y - y_1 = (x - x_1)\frac{dy}{dx}(x_1, y_1)$

$\Rightarrow yy_1^2 - y_1^3 = -x_1^2 x + x_1^3$

$\Rightarrow xx_1^2 + yy_1^2 = x_1^3 + y_1^3 = a^3$ — (I)

$\left[\because (x_1, y_1) \text{ lies on the curve. } x^3 + y^3 = a^3 \right.$

$\therefore x_1^3 + y_1^3 = a^3$

Since the tangent passes through (x_2, y_2)

$\therefore x_2 x_1^2 + y_2 y_1^2 = a^3$ (II) $\cancel{= x_1^3 + y_3^3 = x_2^3 + y_2^3}$

$x_1^3 + y_1^3 = a^3$ — (III)

$\cancel{x_2 x_1^2 + y_2 y_1^2 = x_1^3 + y_1^3}$

x_2

$x_2^3 + y_2^3 = a^3$ — (IV)

(II) — (III)

$x_2 x_1^2 + y_2 y_1^2 = x_1^3 + y_1^3$

$x_2 x_1^2 - x_1^3 = y_1^3 - y_2 y_1^2$

$\Rightarrow x_1^2 (x_2 - x_1) = y_1^2 (y_1 - y_2)$ — (V)

(I) — (IV)

$x_2 x_1^2 + y_2 y_1^2 = x_2^3 + y_2^3$

$x_2 (x_1^2 - x_2^2) = y_2 (y_2^2 - y_1^2)$ — (VI)

B) — VI

$$\frac{x_1^2(x_2-x_1)}{x_2(x_1^2-x_2^2)} = \frac{y_1^2(y_1-y_2)}{y_2(y_2^2-y_1^2)}$$

$$\Rightarrow \frac{x_2(x_1^2-x_2^2)}{x_1^2(x_2-x_1)} = \frac{y_2(y_2^2-y_1^2)}{y_1^2(y_1-y_2)}$$

$$\Rightarrow -\frac{x_2(x_1+x_2)}{x_1^2} = -\frac{y_2}{y_1^2}(y_1+y_2)$$

$$\Rightarrow \frac{x_2^2}{x_1^2} + \frac{x_2}{x_1} = \frac{y_2}{y_1} + \frac{y_2^2}{y_1^2}$$

$$\frac{x_2}{x_1} - \frac{y_2}{y_1} = \frac{y_2^2}{y_1^2} - \frac{x_2^2}{x_1^2}$$

$$= \left(\frac{y_2}{y_1} + \frac{x_2}{x_1}\right)\left(\frac{y_2}{y_1} - \frac{x_2}{x_1}\right)$$

$$\Rightarrow \frac{y_2}{y_1} + \frac{x_2}{x_1} = -1$$

Required res up

Problem 2.

If $p = x\cos\alpha + y\sin\alpha$ touches the Curve

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$$

prove that

$$p^{\frac{m}{m-1}} = (a\cos\alpha)^{\frac{m}{m-1}} + (b\sin\alpha)^{\frac{m}{m-1}}$$

By the question

Solution $\rightarrow$ Equation of the curve is

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$$

Differentiating with respect to x

$$\frac{mx^{m-1}}{a^m} + \frac{my^{m-1}}{b^m}\frac{dy}{dx} = 0$$

$$\frac{x^{m-1}}{a^m} + \frac{y^{m-1}}{b^m}\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{b^m}{a^m}\frac{x^{m-1}}{y^{m-1}}$$

$\therefore$ Equation of tangent at (x_1, y_1) is

$$y - y_1 = -\frac{b^m}{a^m}\frac{x_1^{m-1}}{y_1^{m-1}}(x - x_1)$$

$\Rightarrow a^m y_1^{m-1}(y-y_1) + b^m x_1^{m-1}(x-x_1) = 0$

$\Rightarrow b^m x x_1^{m-1} + b^m x_1^m + a^m y_1^{m-1} y + a^m y_1^m = 0$

$$\Rightarrow \frac{x x_1^{m-1}}{a^m} + \frac{y y_1^{m-1}}{b^m} = \frac{x_1^m}{a^m} + \frac{y_1^m}{b^m} = 1 \quad \text{—(i)}$$

But by the question

$x\cos\alpha + y\sin\alpha = p$ ——— (ii)

Touches the curve at the point (x_1, y_1).

Thus (i) and (ii) represent the same equation. hence they are identical.

Therefore

$$\frac{\frac{x_1^{m-1}}{a^m}}{\cos\alpha} = \frac{\frac{y_1^{m-1}}{b^m}}{\sin\alpha} = \frac{1}{p}$$

$$\Rightarrow \frac{x_1^{m-1}}{a^m\cos\alpha} = \frac{y_1^{m-1}}{b^m\sin\alpha} = \frac{1}{p}$$

$$\Rightarrow x_1^{m-1} = \frac{a^m \cos\alpha}{p}$$

$$\Rightarrow x_1 = \left[\frac{a^m \cos\alpha}{p}\right]^{\frac{1}{m-1}}$$

Similarly $y_1 = \left(\frac{b^m \sin\alpha}{p}\right)^{1/m-1}$

Since (x_1, y_1) lies on the given curve therefore

$$\frac{x_1^m}{a^m} + \frac{y_1^m}{b^m} = 1$$

$$\Rightarrow \frac{\left[\left(\frac{a^m \cos\alpha}{p}\right)^{\frac{1}{m-1}}\right]^m}{a^m} + \frac{\left[\left(\frac{b^m \sin\alpha}{p}\right)^{1/m-1}\right]^m}{b^m} = 1$$

$$\frac{a^{\frac{m^2}{m-1}} \cos\alpha^{\frac{m}{m-1}}}{a^m\, p^{\frac{m}{m-1}}} + \frac{b^{\frac{m^2}{m-1}} (\sin\alpha)^{\frac{m}{m-1}}}{p^{\frac{m}{m-1}}\, b^m} = 1$$

$$\Rightarrow a^{\frac{m^2}{m-1} - m} \cos\alpha^{\frac{m}{m-1}} + b^{\frac{m^2}{m-1} - m} (\sin\alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$$

$$\Rightarrow a^{\frac{\cancel{m^2} - \cancel{m^2} + m}{m-1}} (\cos\alpha)^{\frac{m}{m-1}} + b^{\frac{m^2 - m^2 + m}{m-1}} (\sin\alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$$

$$\Rightarrow a^{\frac{m}{m-1}} (\cos\alpha)^{\frac{m}{m-1}} + b^{\frac{m}{m-1}} (\sin\alpha)^{\frac{m}{m-1}} = p^{\frac{m}{m-1}}$$

Which is required Result

the condition

$x\cos\alpha + y\sin\alpha = p$ should touch the Curve $x^m y^n = a^{m+n}$ is

$$p^{m+n} m^m n^n = (m+n)^{m+n} a^{m+n} \sin^n\alpha \cos^m\alpha$$

Prove that the portion of the Tangent intercepted between the axes is divided at a point of Contact in a Given ratio —

Solution :→ Here Equation of the Curve is

$$x^m y^n = a^{m+n}$$

Now Diff. with respect to x

$$m x^{m-1} y^n + n x^m y^{n-1} \frac{dy}{dx} = 0$$

$$\Rightarrow n x^m y^{n-1} \frac{dy}{dx} = -m x^{m-1} y^n$$

$$\Rightarrow \frac{dy}{dx} = -\frac{m x^{m-1} y^n}{n x^m \cdot y^{n-1}} = -\frac{m}{n}\frac{y}{x}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{m}{n}\frac{y_1}{x_1}$$

Hence the equation of tangent at (x_1, y_1) is

$$y - y_1 = (x - x_1)\left(\frac{dy}{dx}\right)_{x_1, y_1}$$

$$\Rightarrow y - y_1 = (x - x_1)\,\frac{m}{n}\,\frac{y_1}{x_1}$$

$$\Rightarrow n x_1 (y - y_1) = -m y_1 (x - x_1)$$

$$\Rightarrow n x_1 y - n x_1 y_1 = -m y_1 x + m x_1 y_1$$

$$\Rightarrow m y_1 x + n x_1 y = (m+n) x_1 y_1 \quad \text{——} \ (I)$$

But it is given.

$x \cos\alpha + y \sin\alpha = p$ —(II) is also the equation of tangent at (x_1, y_1)

Thus both equation are identical.

$$\frac{m y_1}{\cos\alpha} = \frac{n x_1}{\sin\alpha} = \frac{(m+n) x_1 y_1}{p} \quad \text{——} \ (III)$$

$$\Rightarrow \frac{n x_1}{\sin\alpha} = \frac{(m+n) x_1 y_1}{p}$$

$$\Rightarrow \frac{\sin\alpha}{n \cancel{x_1}} = \frac{p}{(m+n) x_1 y_1}$$

$$\Rightarrow y_1 = \frac{pn}{(m+n)\sin\alpha} \quad \text{and} \quad x_1 = \frac{pm}{(m+n)\cos\alpha}$$

Since (x_1, y_1) lies on the curve

$$x^m y^n = a^{m+n}$$

$$\Rightarrow \quad x_1^m y_1^n = a^{m+n}$$

$$\Rightarrow \frac{(pm)^m (pn)^n}{(m+n)^m \cos^m\alpha \, (m+n)^n \sin^n\alpha} = a^{m+n}$$

$$\Rightarrow p^{m+n} m^m n^n = (m+n)^{m+n} \cos^m\alpha \sin^n\alpha \, a^{(m+n)}$$

Hence the required result

Let the tangent meets the x-axis at A and y-axis at B

then the co-ordinate of A are $\left(\frac{m+n}{m}x_1, 0\right)$

and co-ordinate of B are $\left(0, \frac{m+n}{n}y_1\right)$

Let the point $P(x_1, y_1)$ divides AB in ratio $p:q$ then

$$x_1 = \frac{p\frac{m+n}{m}x_1 + 0}{p+q} \qquad y_1 = \frac{0 + q\frac{m+n}{n}y}{p+q}$$

(using Division Formulae)

$$\Rightarrow \frac{p}{p+q} = \frac{m}{m+n}$$

and $\frac{q}{p+q} = \frac{n}{m+n}$.

$\therefore \frac{p}{q} = \frac{m}{n}$

Which shows the point (x,y) divides the tangent internally between the axes in the constant ratio $\frac{m}{n}$.